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Engineering Notes

Similarity Criteria and Associated Design Procedures for Scaling Solar Sail Systems

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I. Introduction

THE NASA In-Space Propulsion program has promoted the development of solar sail propulsion technology through the development of subsystems, operations tools, analytical and computational models, and ground-based testing. Solar sails can potentially provide low-cost propulsion and operate without the use of propellant, allowing access to non-Keplerian orbits through constant thrust. A solar sail is a gossamer structure: a membrane-based large, lightweight structure. The primary objective of the sail is to convert emitted solar pressure into thrust on a spacecraft. This solar pressure is extremely small, however, on the order of 9 $\mathrm{N/km^2}$ at 1 astronomical unit, resulting in a need for the sail to be very large and lightweight to achieve reasonable accelerations.

The solar sail development process consists of a progression through ground-based testing and flight demonstration to yield a system ready to perform a specific science mission. The science missions envisioned will require sails on the order 2000 to 10, 000 $\mathrm{m^2}$ (or larger); sail size will be dictated by the mass and mission scope of the craft, while flight demonstrations and ground testing will be conducted at significantly smaller sizes [1,2]. For this reason, the ability to understand the process of scalability, as it applies to solar sail system models and test data, is crucial to the advancement of this technology. Because of the significant size of solar sails, the search for scalable sail designs or scaling laws that assist in testing and analyzing solar sail structures are fairly common in the literature. For example, scalable solar sail designs have been proposed by researchers, including Murphy [3], Gaspar et al. [4], Murphy and Murphey [5], and Greshik [6]. Here, the term scalable design seems to imply solar sail designs that retain a basic geometric relationship over various sizes, each tested to validate analytical models [3,4], or architectural design based on a set of sail panels sized to meet engineering requirements and numbered to meet the mission requirements [6]. Alternatively, several researchers develop scaling laws that govern the design of solar sails at larger sizes while maintaining similarity; that is, a model can be built that represents the behavior of a prototype existing at a different scale. For example, Holland et al. [7] observed a set of scaling properties for inflatable structures (boom) based on geometric parameters, while Greschik et al. [8] and Zeiders [9] offer scaling laws for dimensional analysis of solar sail structures.

One way to perform dimensional analysis is through a process of nondimensionalizing a set of governing equations. None of the preceding contributions appear to use this approach. This Note is intended to provide an illustration of deriving the similarity conditions or criteria based on nondimensionalization of the governing equations for a model of the solar sail system. The similarity criteria will define the requirements needed to achieve similarity between a prototype and model solar sail design. The complete number of similarity criteria will be demonstrated through nondimensionalization of the governing equations for the solar sail system following the method of dimensional analysis as demonstrated for a variety of applications in [10–12]. This model will apply to a four-quadrant sail design as presented in [3,6,9]. The model will account for arbitrarily large sail deflection, sail–boom interaction, and the onset of buckling in the boom. The effects of wrinkling in the sail and nonlinear buckling behavior of the boom are two examples of higher-order effects not considered in this model. The results show a set of four independent similarity criteria that must be satisfied. A procedure is offered to demonstrate the use of these similarity criteria to guide the design of a model for ground testing.

II. Method

A schematic of a four-quadrant sail is shown in Fig. 1. The membrane portion of this sail is based on a model that assumes an isotropic sail material of constant thickness h and pressure loading q. This membrane model is correct when bending stiffness is negligible, with respect to extensional stiffness, and the deflection is greater than the material thickness but remains small relative to the sail size (von Kármán plate or Foppl membrane equations). As a result, wrinkling is not considered. The equations of motion for this membrane model of the sail are provided by Timoshenko and Goodier [13] and Mierovitch [14] as

$$F_{xxxx} + 2F_{xxyy} + F_{yyyy} = E[(w_{xy})^2 - w_{xx}w_{yy}] \tag{1}$$

$$q/h + F_{yy}w_{xx} + F_{xx}w_{yy} - 2F_{xy}w_{xy} = \rho w_{tt} \tag{2}$$

where the subscripts x, y, and t denote partial derivatives with respect to the spatial and time parameters. Equation (1) defines compatibility, while Eq. (2) defines Newtonian equilibrium, and F is the force function, w is the sail deflection relative to an undeformed state, E is the sail material modulus, and ρ is the sail density. The force function is defined in terms of the in-plane loads as

$$F_{yy} = N_x/h, \qquad F_{xx} = N_y/h, \qquad F_{yx} = -N_{xy}/h \tag{3}$$

with N representing the stress resultants (in-plane loadings per length).

The boom that supports the sail membrane will be initially considered independent of the sail and then combined with the sail criteria. Based on an Euler–Bernoulli beam model, the equation of motion for a general, uniform boom of length L that considers both flexural and axial rigidity is given as Eq. (4) (see Meirovitch [14], for example):

$$YIv_{zzzz} - Qv_{zz} = \gamma A v_{\tau\tau} \tag{4}$$

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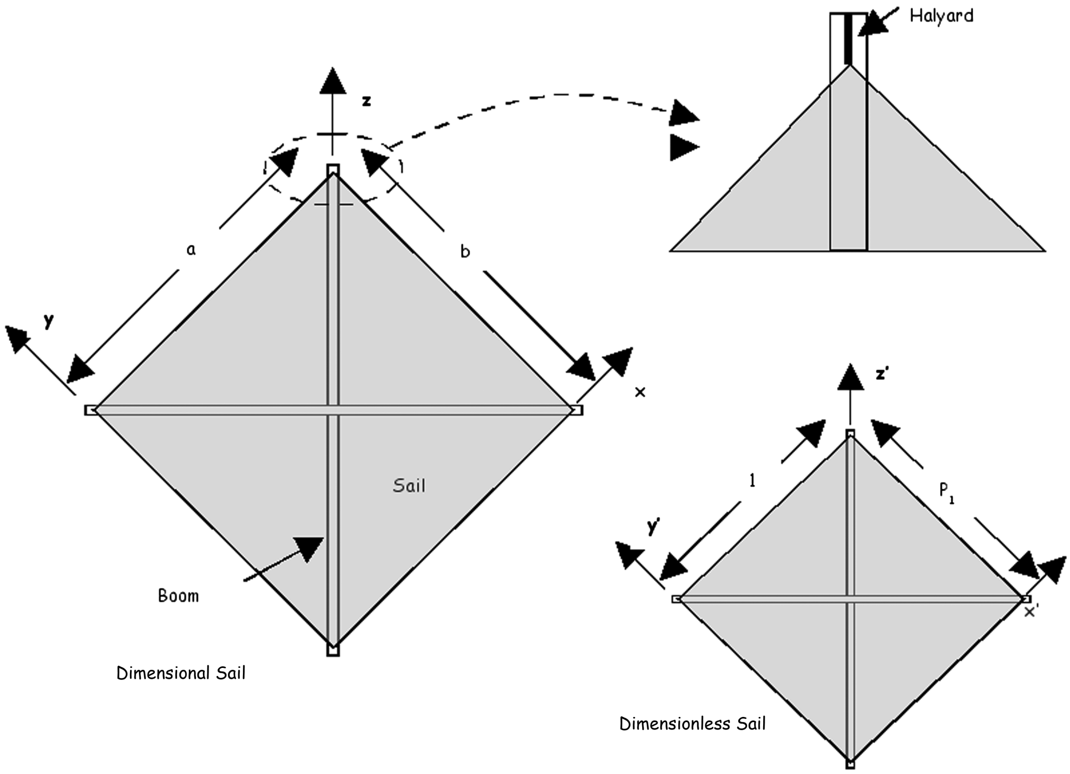

Fig. 1 Four-quadrant sail configuration.

where z is the axial coordinate, Y is the Young's modulus of the boom, I is its area moment of inertia, v is the lateral deflection of the boom, γ is the boom density, A is the boom cross-sectional area, L is the boom length, Q is the axial load on the boom, and τ is the time parameter associated with the boom. The boundary conditions for the boom consist of zero slope and deflection at the boom base, zero bending moment applied at the boom tip, and a known shear force V at the boom tip. This produces the nonzero boundary condition at the boom tip given by Eq. (5):

$$YIv_{zzz} = V \tag{5}$$

The boom and sail parameters are interconnected, resulting in common time and length scales and connectivity at the sail corners and boom tips. This results in the following additional relationships for the model:

$$\sqrt[6]{\frac{\rho^3 h^2 a^4}{Eq_c^2}}t = \sqrt{\frac{\gamma A L^4}{YI}}\tau$$

for equivalent time scales, $L = a\sqrt{a^2 + b^2}$ for equivalent length scales, $Q = -T$ for equal sail and boom in-plane loads, $V = Tw_z$ for equal sail and boom transverse loads, and $v = w$ for equal sail and boom transverse displacements. In these relationships, t and τ are the time scales associated with the sail and boom, respectively, a and b are the sail dimensions, as shown in Fig. 1, and T is the tension in the halyard that connects the boom tip and the sail corner, as shown in Fig. 1. The halyard tension T can be related to the boundary stress resultants, as given in Eq. (6):

$$N_{x,c}, \qquad N_{y,c}, \qquad N_{xy,c} \propto \frac{T}{a} \tag{6}$$

where the subscript c indicates these terms are characteristic stress resultants. Now, a set of dimensionless parameters are obtained that would allow these governing equations to be cast in dimensionless form. Following the techniques found in [11,12], each dimensional term is transformed to a product of a dimensionless term and a nondimensionalizing parameter. The factors of nondimensionalization are not unique but are selected in a manner to identify the primary elements of interest in guiding sail design: for example, geometry, force, deflection, and time. The governing equations are now described with the nondimensional terms, a process that yields the following collection of dimensionless parameters:

$$P_1 = \frac{b}{a} \tag{7}$$

$$P_2 = \frac{N_{c,x}}{\sqrt[3]{Ehq_c^2 a^2}} \tag{8}$$

$$P_3 = \frac{N_{c,y}}{\sqrt[3]{Ehq_c^2 a^2}} \tag{9}$$

$$P_4 = \frac{N_{c,xy}}{\sqrt[3]{Ehq_c^2 a^2}} \tag{10}$$

$$P_5 = \frac{QL^2}{YI} \tag{11}$$

$$P_6 = \left(\frac{\gamma}{\rho}\right)\left(\frac{A}{YI}\right)\left(\frac{Eq_c^2 a^8}{h^2}\right)^{1/3} \tag{12}$$

$$P_7 = \frac{V}{T}\sqrt[3]{\frac{Eh}{q_c a}} \tag{13}$$

$$P_8 = \frac{V}{YI}\sqrt[3]{\frac{Eha^5}{q_c}} \tag{14}$$

where P_i represents the ith dimensionless parameter. Based on these dimensionless parameters, the conditions of similarity between a model and prototype sail can now be defined as

$$P_{i,p} = P_{i,m}, \qquad i = 1, \ldots, 8 \tag{15}$$

with the subscript i referring to the eight criteria, the subscript p identifying the prototype, and the subscript m identifying the model. Here, the model sail represents a baseline sail design of known characteristics, while the prototype sail represents a future design, generally of a much larger size. Equation (15) leads to the criteria that must be met to achieve similarity of the new, larger prototype sail with the current, known model.

III. Results

The first criterion, $C1$, is obtained by relating the first dimensionless parameter [P_1 Eq. (7)] for the model and prototype sail. This criterion describes the requirement of geometric similarity between a model sail (baseline) and prototype sail (new design). The second criterion, $C2$, is derived from the interactions of dimensionless parameters P_2–P_4 [Eqs. (13–15) and the halyard tension [Eq. (6)] between a model and prototype sail with a load connection between the corner of the sail and boom through a halyard. Similarity criterion $C2$ defines the required halyard loading (preload) in a prototype sail based on the sail material properties of a given model, with specified parameters of sail modulus, thickness, pressure loading, and size. Note that for a case of distributed load along the sides of the sail, the similarity criterion $C2$ would be expanded to consider each dimensionless parameter P_2–P_4. Similarity criterion $C3$ is derived by combining the dimensionless parameters P_7 and P_8 over the model and prototype sail along with the condition expressed in $C2$. Similarity criterion $C3$ prescribes the necessary relationship in solar pressure loading for a given sail–boom system design over a model and prototype and can be used for evaluating testing purposes. Finally, similarity criterion $C4$ is derived from the sixth dimensionless parameter (P_6) and the condition expressed in $C3$ and describes geometric similarity requirement on the boom. It should be noted that a similarity criterion derived from dimensionless parameter P_5 is redundant with the criteria $C2$ and $C3$. Similarly, the dimensionless parameters P_7 and P_8 are reduced to a single criterion because of the sail–boom model relationship given in $Q = -T$. Thus, for the model shown with a halyard at the sail corners, a total of four unique similarity criteria exist that must be met in order to achieve similarity between two solar sail systems of a different scale.

A. Proposed Procedure to Use Similarity Criteria in Sail Design

A procedure to use the similarity criteria given in Table 1 is proposed. For the purpose of this procedure, it is assumed that, based on a proposed science mission, a solar sail system design is created that defines the fundamental properties. This sail system is called the prototype, and its defined properties include E_p, h_p, ρ_p, Y_p, $\gamma_p b_p$, A_p, T_p, $q_{c,p}$, and I_p. The objective is to determine the necessary properties of a model sail design of a reduced size that could be tested with available ground-based facilities but would maintain similarity in system response, as defined by the governing equations (1), (2), and (4). Furthermore, due to availability of materials, it is assumed that the sail and boom materials/properties are fixed, giving values for E_m, h_m, ρ_m, Y_m, and γ_m for the model. Furthermore, it is assumed that the available test facilities will define the sail size parameter a. Thus, five design parameters remain to be determined (b_m, A_m, T_m, $q_{c,m}$, and I_m), with four criteria to be satisfied ($C1$–$C4$). Thus, one parameter can be freely selected and the remaining four used to satisfy the design. In general, the similarity criteria $C1$ and $C4$ are used to define additional geometric properties for the sail (b_m from $C1$) and boom (A_m from $C4$). Similarity criteria $C2$ and $C3$ are considered simultaneously to determine the two parameters T_m and I_m, while $q_{c,m}$ is selected to match the available pressure loading in the test environment.

Table 1 Summary of sail–boom system model criteria

Criterion index	Similarity criteria
$C1$	$\frac{b_m}{b_p} = \frac{a_m}{a_p}$
$C2$	$\frac{T_m}{T_p} = \left(\frac{E_m}{E_p}\right)^{1/3}\left(\frac{h_m}{h_p}\right)^{1/3}\left(\frac{q_{c,m}}{q_{c,p}}\right)^{2/3}\left(\frac{a_m}{a_p}\right)^{5/3}$
$C3$	$\left(\frac{q_{c,m}}{a_{c,p}}\right) = \left(\frac{Y_m I_m}{Y_p I_p}\right)^{3/2}\left(\frac{E_p}{E_m}\right)^{1/2}\left(\frac{h_p}{h_m}\right)^{1/2}\left(\frac{a_p}{a_m}\right)^{11/2}$
$C4$	$\left(\frac{A_m}{A_p}\right)^{3/2} = \left(\frac{\rho_m}{\rho_p}\right)^{3/2}\left(\frac{\gamma_p}{\gamma_m}\right)^{3/2}\left(\frac{h_m}{h_p}\right)^{3/2}\left(\frac{a_m}{a_p}\right)^{3/2}$

Table 2 Sail system parameters for four models

Sail system parameter	Model 1	Model 2	Model 3	Model 4
a, m	10	20	30	40
E, Pa	$3.9(10^9)$	$3.9(10^9)$	$3.9(10^9)$	$3.9(10^9)$
h , m	$5.58(10^{-6})$	$5.58(10^{-6})$	$5.58(10^{-6})$	$5.58(10^{-6})$
ρ, kg/m^3	1392.5	1392.5	1392.5	1392.5
Y, Pa	$7.854(10^{-5})$	$7.854(10^{-5})$	$7.854(10^{-5})$	$7.854(10^{-5})$
q, Pa	$9.0(10^{-6})$	$9.0(10^{-6})$	$9.0(10^{-6})$	$9.0(10^{-6})$
b, m	10	20	30	40
I, m^4	$7.854(10^{-5})$	$9.974(10^{-5})$	$4.411(10^{-5})$	$1.267(10^{-5})$
T, N	1	3.18	6.24	10.08

Table 3 Sail system response results

Response	Model 1	Model 2	Model 3	Model 4
w (predict), m	$2.47(10^{-4})$	$6.23(10^{-4})$	$10.7(10^{-4})$	$15.7(10^{-4})$
w (numeric), m	$2.47(10^{-4})$	$6.26(10^{-4})$	$10.8(10^{-4})$	$15.8(10^{-4})$
w % error	0	0.5	0.6	0.6
v (predict), m	$1.70(10^{-9})$	$4.28(10^{-9})$	$7.36(10^{-9})$	$10.8(10^{-9})$
v (FEA), m	$1.70(10^{-9})$	$4.27(10^{-9})$	$7.35(10^{-9})$	$10.8(10^{-9})$
v % error	0	0.3	0.2	0.4

B. Examples of Sail Scaling

A numerical example is provided to demonstrate the expected response over a series of four similar sail systems of four different sizes: 10, 20, 30, and 40 m. Table 2 shows the properties of the four sail systems that are selected to satisfy the similarity criteria summarized in Table 1. Each sail is modeled numerically using the commercial finite element analysis package, ANSYS [15]. The behavior of each sail, as predicted by the nondimensional models, is compared with the behavior predicted by ANSYS (and shown on Table 3) for deflection at the sail w measured at the centroid and deflection at the boom tip v.

Table 3 shows good agreement between the scaling rules developed in this Note and the numerical predictions. Other comparisons (not shown for the sake of brevity) involving sail-only and boom-only systems yielded similar levels of agreement.

IV. Conclusions

Scaling laws applicable to a class of solar sail systems are developed. The scaling laws consist of four similarity criteria that must be satisfied in selecting primary sail system parameters to achieve similarity between a model and a prototype sail. These similarity criteria can be used to guide the design of scaled experimental tests for solar sails and in analyzing the results from such tests. The similarity criteria are derived directly from the governing equations for a solar sail system model. Note that, since the focus of the study is on obtaining similarity criteria global shape behavior of the sail system, the present analysis ignores effects such as wrinkling and nonlinear buckling behavior. Improved correlation could potentially be gained by enhancing the approach with advanced analytical approaches that have previously been employed to study membrane surface profiles on both the global and local scales.

A procedure is demonstrated to illustrate the process of using the similarity criteria to guide the design process: in particular, in choosing a number of the fundamental sail system parameters. Through this procedure, it was demonstrated that, for a fairly common situation in which many of the design parameters are defined by other constraints on material and facilities, it is possible and reasonable to create scaled sail models around the restrictions of a ground-based experiment. In particular, the example demonstrates that unique sail properties and gravitational loads may be accommodated in the static design.

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